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Decimation Transformations in Lattice Systems of Continuous Spins

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Decimation renormalization transformations are investigated for systems of continuous spins. The usual arguments against decimation can be avoided by considering products of decimation and spin scaling transformations. With the simple local types of spin scaling normally used for continuous spins, even these product transformations will have no fixed points for lattice dimension greater than one. A Gaussian fixed point for one-dimensional models with short range (but not only nearest neighbor) interactions is exhibited. A series of scaling transformations of increasing generality is investigated. It is found that a product of a nonlocal spin scaling transformation and a decimation will produce the usual fixed points, but that this type of product transformation is effectively much more a "block"-type transformation than a pure decimation.

KEY WORDS: Renormalization group; decimation; spin scaling; continuous spins; fixed points.

1. INTRODUCTION

Several authors[1-6] have investigated the critical behavior of Ising spin systems using the decimation type renormalization transformations. All of these authors note that these transformations can have a fixed point on the critical surface only if the exponent η satisfies $d - 2 + \eta = 0$, that is, only if the order parameter correlation function is not zero at large distances. One-dimensional Ising models have, in a sense, a critical point at $T = 0$ which satisfies the above condition, and the properties of several of these models have been rederived[5] using the decimation transformation with no approximations. Two-dimensional Ising systems have been investigated[1,2] using approximate decimation transformations. The approximate transformations show fixed points and good numerical results are obtained for

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some exponents even though these models do not satisfy the $d - 2 + \eta = 0$ condition. Since the fixed points must be produced by the approximations, it is not clear why these methods yield good numerical results.

The usual argument, that decimation transformations and critical fixed-point behavior are incompatible when $d - 2 + \eta \neq 0$, fails if the decimation is combined with some type of spin scaling transformation and provided the spin scaling factor is chosen properly. Transformations of this kind have been used,[3,6] with approximations, to examine the critical behavior of two-dimensional Ising systems. Again, the approximate transformations show fixed points and give good numerical values for the critical exponents. The approximate transformations, however, have fixed points for a range of values of the spin scaling parameter, whereas the exact transformation can have a critical fixed point only for a particular choice of the spin scaling parameter. It is again not clear to what extent the approximate results are characteristic of the exact transformations. In this case, however, improvement[3] in the approximations narrow the allowed range of spin scaling factors and suggest that the correct factor would be determined by the exact transformation.

There are no exact calculations using these transformations on models of more than one dimension. This paper considers decimation with spin scaling applied to continuous spin systems and, in particular, the question of whether the exact transformations can produce the fixed points[7,8] which occur so naturally in the block-type transformations. The answer to this question is a qualified no. In one dimension there are Gaussian fixed points, but in higher dimensions, if the spin scaling procedures used are the "usual" ones for continuous spin models, there are no fixed points on the critical surface. Other types of spin scaling are considered and difficulties with them are noted. All of the transformations considered are of the type usually termed linear, that is, correlations of a given order are related by the transformations only to correlations of the same order.

2. DECIMATION AND SPIN RESCALING

The decimation type of transformation is always effected by dividing a lattice into several sublattices and then integrating over the degrees of freedom associated with some of the sublattices. There are many possible choices of the original lattice and for the decomposition into sublattices. The choices made here are for simplicity. The conclusions do not appear to depend on these particular choices, though this is not proven here.

Consider a d-dimensional simple cubic lattice L of unit spacing with lattice points given by vectors $\mathbf{n} = (n_1, n_2,..., n_d)$ with integer components. Let $x_\mathbf{n}$ be continuous variables associated with each lattice point $\mathbf{n}$. The Hamiltonian $\mathscr{H}_n[x]$ is a function of all the $x_\mathbf{n}$ for which $\mathbf{n}$ lies in the cube, containing N^d lattice points, specified by $-N/2 \leqslant n_i < N/2$ for $1 \leqslant i \leqslant d$. This lattice can be subdivided into 2^d simple cubic lattices, each of two-unit spacing, by allowing each component n_i to range over either even or odd integers. Let L_e be the set of lattice points whose every component is even and let L_c be all other lattice points of L ($L = L_e \cup L_c$). The decimation transformation will consist of integrating over all $x_{\mathbf{n}}$ associated with L_c and relabeling the remaining variables (those of L_e) by the substitution $x'_{\mathbf{n}} = x_{\mathbf{n}/2}$ so that they are associated with a lattice of unit spacing. The new Hamiltonian will have $N' = N2^{-d}$ variables. We indicate the transformation by D and write it, symbolically, as

$$D: \quad \exp(\mathscr{H}'_{N'}[x']) = \int_{\mathbf{n}\to\mathbf{n}/2} \exp(\mathscr{H}_N[x]) \prod_{\mathbf{n}\in L_c} dx_{\mathbf{n}} \tag{1}$$

The transformation leaves the partition function invariant; $Z(\mathscr{H}'_{N'}) = Z(\mathscr{H}_N)$. The two-point correlation functions $\Gamma(\mathbf{n}, \mathscr{H}) = \langle x_{\mathbf{m}} x_{\mathbf{m}+\mathbf{n}} \rangle_{\mathscr{H}}$ and $\Gamma(\mathbf{n}, \mathscr{H}') = \langle x'_{\mathbf{m}} x'_{\mathbf{m}+\mathbf{n}} \rangle_{\mathscr{H}'}$ are related by

$$D: \qquad \Gamma(\mathbf{n}, \mathscr{H}') = \Gamma(2\mathbf{n}, \mathscr{H}), \qquad \mathbf{n} \in L \tag{2}$$

if $\mathscr{H}$ is a fixed-point Hamiltonian, then $\mathscr{H} = \mathscr{H}' = \mathscr{H}^*$ and (2) becomes $\Gamma(\mathbf{n}, \mathscr{H}^*) = \Gamma(2\mathbf{n}, \mathscr{H}^*)$, which is inconsistent[3] with the usually assumed asymptotic form

$$\Gamma \propto |\mathbf{n}|^{-d+2-\eta} \tag{3}$$

for large $\mathbf{n}$ unless $-d + 2 - \eta = 0$. This condition is not satisfied for most systems of interest and therefore the decimation transformation has no fixed point on their critical surface.

The above argument can be circumvented by combining the decimation transformation with a transformation[8] S which rescales *all* variables $x_{\mathbf{n}}$. Perhaps the simplest example is the transformation S_1 which just multiplies each $x_{\mathbf{n}}$ by a factor β,

$$S_1: \qquad \exp(\mathscr{H}_N'[x']) = \int \prod_{\mathbf{n}\in L} \delta(x'_{\mathbf{n}} - \beta x_{\mathbf{n}}) \exp(\mathscr{H}_N[x]) \prod_{\mathbf{n}\in L} dx_{\mathbf{n}} \tag{4}$$

It is straightforward that under S_1, $Z(\mathscr{H}_N') = Z(\mathscr{H}_N)$ and

$$S_1: \qquad \Gamma(\mathbf{n}, \mathscr{H}') = \beta^2 \Gamma(\mathbf{n}, \mathscr{H}), \qquad \mathbf{n} \in L \tag{5}$$

Consider now the product transformation $S_1 D$. It will leave the partition function invariant. The correlation functions transform as (2) followed by (5):

$$S_1 D: \qquad \Gamma(\mathbf{n}, \mathscr{H}') = \beta^2 \Gamma(2\mathbf{n}, \mathscr{H}), \qquad \mathbf{n} \in L \tag{6}$$

This transformation is consistent with a fixed point and the asymptotic form (3) provided $\beta^2 = 2^{d-2+\eta}$ (β is 2^{2d} times the usual spin scaling factor[7,8]). Now, however, another difficulty appears. Since (6) is valid for $\mathbf{n} = 0$, it requires that at a fixed point $\mathscr{H}^*$, $\Gamma(0, \mathscr{H}^*) = \beta^2\Gamma(0, \mathscr{H}^*)$. But $\Gamma(0, \mathscr{H}^*) \neq 0$ (it is the average of a positive quantity); therefore $\beta^2 = 1$, which again requires $d - 2 + \eta = 0$ if there is to be a fixed point on the critical surface.

The difficulty in the above transformation is that it scales Γ at $\mathbf{n} = 0$ by the same factor as for large $\mathbf{n}$. This is not the case with the spin scaling transformation[8] S_2 given by

$$S_2: \qquad \exp(\mathscr{H}_N'[x']) = \left(\frac{\alpha}{2\pi}\right)^{Nd/2} \int \prod_{\mathbf{n}\in L} \exp[-\tfrac{1}{2}\alpha(x_{\mathbf{n}}' - \beta x_{\mathbf{n}})^2] \times \exp(\mathscr{H}_N[x]) \prod_{\mathbf{n}\in L} dx_{\mathbf{n}} \tag{7}$$

The partition function is invariant under S_2 and the correlation function transforms as[8]

$$S_2: \qquad \Gamma(\mathbf{n}, \mathscr{H}') = \beta^2\Gamma(\mathbf{n}, \mathscr{H}) + \alpha^{-1}\delta_{\mathbf{n},0} \tag{8}$$

and under transformation S_2D as

$$S_2D: \qquad \Gamma(\mathbf{n}, \mathscr{H}') = \beta^2\Gamma(2\mathbf{n}, \mathscr{H}) + \alpha^{-1}\delta_{\mathbf{n},0} \tag{9}$$

The large-$\mathbf{n}$ behavior of Γ again requires the $\beta = 2^{d-2+\eta}$ if there is to be a fixed point. But for $\mathbf{n} = 0$ and $\mathscr{H} = \mathscr{H}' = \mathscr{H}^*$, (9) yields $\Gamma(0, \mathscr{H}^*) = [\alpha(1 - \beta^2)]^{-1}$. This equation cannot be satisfied. $\Gamma(0, \mathscr{H}^*)$ is positive and S_2 can be defined only for positive α, so that β^2 must be less than 1. But $\beta^2 = 2^{d-2+\eta}$ is greater than 1 because the correlations vanish as $\mathbf{n} \to \infty$ at the usual critical point.

Neither of transformations S_1D or S_2D will have fixed points corresponding to those of the transformations usually used with continuous systems.[7,8] The one- and two-dimensional Gaussian models are excepted from this conclusion. For these models $\Gamma(\mathbf{n}, \mathscr{H}) \to \infty$ as $\mathscr{H}$ approaches the critical surface, so the above arguments are inappropriate. Before considering the possibility of more general scaling transformations S, it is worth considering the Gaussian model in more detail.

3. GAUSSIAN MODELS

The simplest fixed points, under the usual transformations, for continuous systems are the Gaussian fixed points. The above arguments indicate that even these cannot occur, at least when the correlation functions are well defined, for the SD transformations. To examine this more closely, consider the space of Hamiltonians of the form

$$\mathscr{H} = -\tfrac{1}{2}\sum_{\mathbf{m},\mathbf{n}} J(\mathbf{n})x_{\mathbf{m}}x_{\mathbf{m}+\mathbf{n}} \tag{10}$$

where the interactions $J(\mathbf{n})$ are such that $\mathscr{H}[x] \leqslant 0$ for all values of the $x_{\mathbf{n}}$.

Assuming the usual periodic boundary conditions, these Hamiltonians are diagonalized by the transformations

$$x_{\mathbf{q}} = \sum_{-N/2 \le n_i \le N/2} [\exp(-i\mathbf{q}\cdot\mathbf{n})]x_{\mathbf{n}}, \qquad x_{\mathbf{n}} = N^{-d} \sum_{0}^{2\pi} [\exp(i\mathbf{q}\cdot\mathbf{n})]x_{\mathbf{q}} \tag{11}$$

where each component of $\mathbf{q}$ varies from 0 to 2π in steps of $2\pi/N$. The Hamiltonian $\mathscr{H}$ takes the form

$$\mathscr{H} = -\tfrac{1}{2}N^{-d} \sum_{0}^{2\pi} \mathscr{J}(\mathbf{q})x_{\mathbf{q}}x_{-\mathbf{q}}, \qquad \mathscr{J}(\mathbf{q}) = \sum_{\mathbf{n}} [\exp(-i\mathbf{q}\cdot\mathbf{n})]J(\mathbf{n}) \tag{12}$$

and the correlations are

$$\langle x_{\mathbf{q}}x_{\mathbf{q}'}\rangle = \delta_{\mathbf{q}+\mathbf{q}',0}N^{d}/\mathscr{J}(\mathbf{q}) \tag{13}$$

The Hamiltonians of interest are those for which $\mathscr{J}(q) > 0$, $q \neq 0$, and $\mathscr{J}(0) \geqslant 0$. The critical surface is defined by $\mathscr{J}(0) \equiv r_0 = 0$. It is also assumed, as usual, that for small $\mathbf{q}$, $\mathscr{J}(\mathbf{q}) \simeq r_0 + zq^2 + \cdots$. The correlation function $\Gamma(\mathbf{n}, J)$ is related to $\mathscr{J}(\mathbf{q})$ by

$$\Gamma(\mathbf{n}, J) = N^{-d} \sum_{0}^{2\pi} [\exp(i\mathbf{q}\cdot\mathbf{n})]/\mathscr{J}(\mathbf{q}) \tag{14}$$

$$\mathscr{J}^{-1}(\mathbf{q}) = \sum_{-N/2 \le n_i < N/2} [\exp(-i\mathbf{q}\cdot\mathbf{n})]\Gamma(\mathbf{n}, J) \tag{15}$$

as long as $r_0 \neq 0$ so that $\mathscr{J}^{-1}(\mathbf{q})$ is nonsingular. The sum in (14) becomes an integral, as $\mathbf{n} \to \infty$, which is finite for $d \geqslant 3$ but not for $d \leqslant 2$ as $r_0 \to 0$.

It is straightforward to check that the transformations S_1D and S_2D transform Gaussian Hamiltonians into Gaussian Hamiltonians. The transformation of the Hamiltonian, that is, $\mathscr{J}(\mathbf{q})$, is most easily found from the transformation of $\Gamma(\mathbf{n}, J)$. Consider the transformation S_1D given by (6). Putting $\Gamma(\mathbf{q}, J) = 1/\mathscr{J}(\mathbf{q})$ and writing (14) for $\Gamma(\mathbf{n}, J')$ gives

$$\Gamma(\mathbf{n}, J') = \left(\frac{N}{2}\right)^{-d} \sum_{0}^{2\pi} [\exp(i\mathbf{q}\cdot\mathbf{n})]\Gamma(\mathbf{q}, J') \tag{16}$$

where in the sum, each component of $\mathbf{q}$ varies in steps of $4\pi/N$, since after the SD transformation the system contains $(N/2)^d$ lattice sites. Then, using (14) for $\Gamma(2\mathbf{n}, J)$ gives

$$\Gamma(2\mathbf{n}, J) = N^{-d} \sum_{0}^{2\pi} [\exp(i\mathbf{q}\cdot 2\mathbf{n})]\Gamma(\mathbf{q}, J) = N^{-d} \sum_{0}^{4\pi} [\exp(i\mathbf{q}'\cdot\mathbf{n})]\Gamma(\mathbf{q}'/2, J)$$

In the second sum the components of $\mathbf{q}'$ vary in steps of $4\pi/N$. Let $2\pi\mathbf{K} = 2\pi(K_1, K_2,..., K_d)$, $K_i = 0, 1$. The range of the second sum can be reduced to 0–2π by adding all possible $2\pi\mathbf{K}$ to $\mathbf{q}'$ and since $\mathbf{n}\cdot\mathbf{K} = 1$; we have

$$\Gamma(2\mathbf{n}, J) = N^{-d} \sum_{q_i=0}^{2\pi} \sum_{K_i=0}^{1} [\exp(i\mathbf{q}\cdot\mathbf{n})]\Gamma\left(\frac{\mathbf{q} + 2\pi K}{2}, J\right) \tag{17}$$

Now (16) and (17) with (6) imply

$$\Gamma(\mathbf{q}, J') = \beta^2 2^{-d} \sum_{K_i=0}^{1} \Gamma\left(\frac{\mathbf{q} + 2\pi\mathbf{K}}{2}, J\right) \tag{18}$$

and iterating this equation p times yields

$$\Gamma(\mathbf{q}, J^{(p)}) = \beta^{2p} 2^{-dp} \sum_{-2^{p-1} < K_i < 2^{p-1}} \Gamma\left(\frac{\mathbf{q} + 2\pi\mathbf{K}}{2^p}, J\right) \tag{19}$$

where the periodicity of $\Gamma(\mathbf{q}, J)$ has been used to change the limits of summation from $0 \leqslant K_i < 2^p$ to $-2^{p-1} \leqslant K_i < 2^{p-1}$. Off the critical surface ($r_0 \neq 0$) Γ is nonsingular and the sum (19) approaches an integral as $p \to \infty$. A family of high-temperature fixed points, corresponding to noninteracting spins, is reached for the choice $\beta = 1$ and is given by

$$\frac{1}{\mathscr{J}^*(\mathbf{q})} = \Gamma(\mathbf{q}, J^*) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{d\mathbf{K}}{\mathscr{J}(\mathbf{K})}.$$

On the critical surface ($r_0 = 0$), $\Gamma(\mathbf{q}, J)$ has a $1/zq^2$ singularity as $\mathbf{q} \to 0$. The integral (in the thermodynamic limit) in (14) is divergent at small $\mathbf{q}$ for one- or two-dimensional systems and therefore $\Gamma(\mathbf{n}, J)$ is infinite. $\mathscr{J}(\mathbf{q})$ [and $\Gamma(\mathbf{q}, J)$] exists, however, and still transforms according to (18) since neither the decimation nor the scaling transformation depends on $J(\mathbf{n})$ in a singular way near the critical surface. The transformation law (18) can also be derived without reference to $\Gamma(\mathbf{n}, J)$ by introducing separate coordinates $x_{\mathbf{q},\alpha}$, $\alpha = 1,..., 2^d$, for each sublattice and then integrating over all sublattices but one by a series of completing the square type arguments. This derivation, though simple in concept, is tedious in execution and will not be given.

For $r_0 = 0$, $\Gamma(\mathbf{q}, J)$ can be written $\Gamma(\mathbf{q}, J) = 1/zq^2 + F(\mathbf{q})$, where $F(\mathbf{q})$ is not singular. If this is put in (19), the sum over the nonsingular part approaches an integral, so (19) becomes, for large p,

$$\Gamma(\mathbf{q}, J^{(p)}) = z^{-1}\beta^{2p} 2^{(2-d)p} \sum_{-2^{p-1} \leq K_i < 2^{p-1}} |\mathbf{q} + 2\pi\mathbf{K}|^{-2} + \beta^{2p}(2\pi)^{-d} \int_{-\pi}^{\pi} F(\mathbf{K})\, d\mathbf{K} \tag{20}$$

For $d = 1$ the sum in (20) is convergent and the first term will approach a nontrivial limit for large p if $\beta^2 = 1/2$. With this β the second term vanishes for large p so

$$\Gamma(\mathbf{q}, J^*) = \frac{1}{z} \sum_{K=-\infty}^{+\infty} \frac{1}{|q + 2\pi K|^2}$$

The sum of this series is[9] $[2\sin(q/2)]^{-2}$. The fixed points for $d = 1$ are therefore

$$\mathscr{J}^*(q) = 4z \sin^2(q/2) \tag{21}$$

which means the fixed-point interactions $J^*(n)$ are nearest neighbor and of strength z. So, in one dimension, every Hamiltonian on the critical surface approaches a nearest neighbor fixed point of the same z under iteration of the S_1D transformation.

For $d = 2$, however, (20) will not yield a nontrivial fixed point. The sum in (20) diverges (slowly) as $p \to \infty$, so no finite limit is approached if $\beta \geqslant 1$. But if $\beta < 1$, $\beta^{2p} \to 0$ so rapidly that the limit of (20) is zero. Therefore, the transformation S_1D has Gaussian fixed points, of the usual kind, for $d = 1$ but none for $d \geqslant 2$. The same is true, by similar arguments, for S_2D. Generalizations of these transformations that may show fixed points are considered next.

4. OTHER SCALING TRANSFORMATIONS

A natural generalization of the transformations S_1 and S_2 is

$$S_3: \qquad \exp(\mathscr{H}_N'[x']) = \int \prod_{\mathbf{n}\in L} s(x_{\mathbf{n}}' - \beta x_{\mathbf{n}}) \exp(\mathscr{H}_N[x]) \prod_{\mathbf{n}\in L} dx_{\mathbf{n}} \tag{22}$$

where $s(y)$ is an even function and normalized so $\int s(y)\,dy = 1$. It is easy to show that the partition function is invariant under S_3 and that the correlation function still transforms as (8) with $\alpha^{-1} = \int y^2 s(y)\,dy$. If $s(y) > 0$, then $\alpha > 0$ and the argument of Section 2 shows there can be no fixed point of the transformations S_3D. If $s(y)$ is not everywhere positive, then it is possible that $\alpha < 0$, voiding the argument of Section 2. Then, however, it is not clear from (22) under what conditions the transformed Hamiltonian $\mathscr{H}_N'[x']$ will be real. This is reminiscent of a similar difficulty known[10] to occur for certain linear renormalization transformations of discrete spin systems. Another difficulty with S_3 is that, except for special choices of $s(y)$, it does not transform Gaussian Hamiltonians into Gaussian Hamiltonians and hence the existence of a Gaussian fixed point is difficult to investigate.

The scaling transformations S_3 are local, in the sense that the kernel of the integral transform (22) is a product of factors each of which couples each new variable to the old one at only the same lattice site. A nonlocal generalization of (22) is

$$S_4: \qquad \exp(\mathscr{H}_N[x']) = \int \prod_{\mathbf{n}\in L} s\left(x_{\mathbf{n}}' - \sum_{\mathbf{m}} \beta(\mathbf{m}) x_{\mathbf{n}+\mathbf{m}}\right) \times \exp(\mathscr{H}_N[x]) \prod_{\mathbf{n}\in L} dx_{\mathbf{n}} \tag{23}$$

where the new variable $x_{\mathbf{n}}$ is related to a weighted average of the $x_{\mathbf{n}+\mathbf{m}}$ at neighboring sites. The choice $\beta(\mathbf{m}) = \delta_{\mathbf{m},0}$ reduces (23) to (22). If again

$\int s(y)\,dy = 1$ and s is even, then the partition function is invariant under (23) and the correlation function transforms as

$$\Gamma(\mathbf{n}, \mathscr{H}') = \sum_{\mathbf{k},\mathbf{l}} \beta(\mathbf{k})\beta(\mathbf{l})\Gamma(\mathbf{n} + \mathbf{l} - \mathbf{k}, \mathscr{H}) + \delta_{\mathbf{n},0}\alpha^{-1}$$

This scaling transformation can be combined with the decimation as either S_4D or DS_4, which are not the same. These transformations have, however, essentially the same behavior under repeated iterations, so we shall consider only DS_4. The correlation functions transform as

$$DS_4: \qquad \Gamma(\mathbf{n}, \mathscr{H}') = \sum_{\mathbf{k},\mathbf{l}} \beta(2\mathbf{k})\beta(2\mathbf{l})\Gamma(2\mathbf{n} + 2\mathbf{l} - 2\mathbf{k}, \mathscr{H}) + \delta_{\mathbf{n},0}\alpha^{-1} \tag{24}$$

Now, in contrast to (9), the new correlation function at $\mathbf{n} = 0$ is not related to the old one at only $\mathbf{n} = 0$. Thus the argument forbidding a fixed point fails even for $\alpha > 0$. Furthermore, it seems reasonable to expect DS_4 to be similar to the block type of transformation,[7,8] since the scaling transformation S_4 "smears" the spin over some distance determined by the range of $\beta(\mathbf{m})$ and the decimation removes some of these smeared spins. Also, with an appropriate choice of $\beta(\mathbf{m})$, (24) is very similar to the transformation of the correlations under the block-type transformations. Since the block-type transformations have fixed points, it might be expected that DS_4 does also. This can be explicitly demonstrated for the Gaussian fixed points by choosing, in (23), $s(y) = (\alpha/2\pi)^{1/2}e^{-\alpha y^2}$. With this choice (23) will transform a Gaussian Hamiltonian into a Gaussian Hamiltonian and the transformation of the Hamiltonians can be found from (15) and (23). An argument similar to that leading to (18) yields

$$\Gamma(\mathbf{q}, J') = 2^{-d} \sum_{K_i=0}^{1} \left|\beta\left(\frac{\mathbf{q} + 2\pi\mathbf{K}}{2}\right)\right|^2 \Gamma\left(\frac{\mathbf{q} + 2\pi\mathbf{K}}{2}, J\right) \tag{25}$$

Suppose $\beta(\mathbf{m})$ is chosen to average over a cube of unit side length, for example, $\beta(\mathbf{m}) = \beta$ for all $\mathbf{m}$ with components $m_i = 0, 1$ and $\beta(\mathbf{m}) = 0$ otherwise. Then

$$\beta(\mathbf{q}) = \sum_{\mathbf{m}} [\exp(-i\mathbf{q}\cdot\mathbf{m})]\beta(\mathbf{m}) = \beta \sum_{m_j=0}^{1} \exp(-i\mathbf{q}\cdot\mathbf{m}) = \prod_{j=1}^{d} [1 + \exp(-iq_j)]$$

and

$$|\beta(q)|^2 = \prod_{j=1}^{d} \frac{\sin^2 q_j}{\sin^2(q_j/2)}$$

With this choice of $\beta(\mathbf{m})$, (25) is identical to the block transformation used by Bell and Wilson[8] and shown by them to have Gaussian fixed points.

In conclusion, it appears that, for continuous spins, decimation combined with local spin rescaling of the usual types will produce fixed points only for one-dimensional lattices. Decimation with nonlocal spin rescaling may produce fixed points in any dimension, but such transformations are, at least in some simple cases, equivalent to the block-type transformations.

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